MAT401 **Due Date : Tuesday, May 24, 11:59 pm**
SUMMER 2022 **Submission : Crowdmark**

Problem Set 1

As in lectures, by rings we shall mean a commutative ring with identity, unless otherwise specified. Textbook questions are from "Galois Theory" (2nd edition) by Joseph Rotman.

1. Textbook Exercise 9.
2. (a) Let $R_1, R_2, \cdots, R_n$ be rings. Construct the direct sum

$$R_1 \oplus R_2 \oplus \cdots \oplus R_n \;=\; \{(r_1, r_2, \cdots, r_n) \mid r_i \in R_i\}$$

Define componentwise addition and multiplication on this set. That is,

$$(r_1, r_2, \cdots, r_n) \oplus (s_1, s_2, \cdots, s_n) \;=\; (r_1 + s_1, r_2 + s_2, \cdots, r_n + s_n) \quad \text{and}$$
$$(r_1, r_2, \cdots, r_n) \odot (s_1, s_2, \cdots, s_n) \;=\; (r_1 \cdot s_1,\ r_2 \cdot s_2, \cdots,\ r_n \cdot s_n)$$

Show that $(R_1 \oplus R_2 \oplus \cdots \oplus R_n\ ,\ \oplus\ ,\odot)$ is a ring.

(b) Prove or give counter example : If D_1 and D_2 are domains, then so is $D_1 \oplus D_2$.

3. Show that R is a domain if and only if is $R[x]$ is a domain.
4. (a) Recall that we denote degree of a polynomial $f(x)$ by $\partial(f)$. Show that if R is a domain, then for any two non-zero polynomials $f(x),\ g(x) \in R[x]$,

$$\partial(fg) = \partial(f) + \partial(g)$$

(b) Give a counter example to show that the above statement need not hold if R is not a domain.

5. (a) Show that every non-zero element in $\mathbb{Z}_n$ is either a unit or a zero-divisor.

(b) Give an example of a ring and a non-zero element in it, which is neither a zero-divisor nor a unit.

Note : For 5.(a) , if needed you may use Exercise 11 of the Textbook, without a proof.