MAT401 **Due Date : Monday, June 6, 11:59 pm**

SUMMER 2022 **Submission : Crowdmark**

Problem Set 2

As in lectures, by rings we shall mean a commutative ring with identity, unless otherwise specified. Textbook questions are from "Galois Theory" (2nd edition) by Joseph Rotman.

1. Let D be an Euclidean domain with respect to a function $f : D \setminus \{0\} \to \mathbb{Z}_{\geq 0}$. Suppose f satisfies the following extra condition : $f(a) \leq f(ab)$ for all $a, b \in D \setminus \{0\}$.

   (a) Show that u is a unit in D iff $f(u) = f(1)$

   (b) If $a = ub$, where u is a unit in D, then $f(a) = f(b)$.

   (c) Suppose f also satisfies $f(a+b) \leq \max\{f(a), f(b)\}$ for all $a, b \in D$, $a + b \neq 0$. Show that the quotient and the remainder in division algorithm is unique under this extra assumption on f.

   Remark : Such a function in an Euclidean domain is sometimes called a "Norm". Some textbooks include these properties within the definition of a Euclidean Domain.

2. (a) Textbook exercise 24. [If $\phi : R \to S$ is an ...]

   (b) Textbook exercise 27. [If $a \in R$ is a unit ...]

3. Let R be a subring of a field F. The two parts of this problem, is about showing that $\text{Frac}(R)$ is the smallest field containing R, in two different ways.

   (a) For any other subfield L of F, such that $R \subseteq L \subseteq F$, show that there exists an injective ring homomorphism $\phi : \text{Frac}(R) \to L$.

   (b) Textbook exercise 25.

   [$\text{Frac}(R) \cong \bigcap K_\alpha$, for all subfields K_α of F containing R.]

4. Textbook exercise 32. [Let u be a unit ...]

Bonus problem in next page!

Policy : Solving this question will get you a 2% course bonus, even if this PS is dropped while calculating your final grade. Of course, the maximum possible course grade is 100. Note that this question will be marked either 0 or 2, no partial marks. I encourage all of you to try it, but just to save time, please do not submit a partial answer or one that you're completely uncertain of.

Bonus Problem (2%)

Let K be a field and $K^\times$ denote $K \setminus \{0\}$. A **discrete valuation** on K is a function $\nu : K^\times \to \mathbb{Z}$ satisfying the following properties :

(i) $\nu(ab) = \nu(a) + \nu(b)$ for all $a, b \in K^\times$
(ii) ν is surjective.
(iii) $\nu(x + y) \geq \min\{\nu(x),\ \nu(y)\}$ for all $x, y \in K^\times,\ \ x + y \neq 0$

The set $R_\nu := \{x \in K \mid \nu(x) \geq 0\} \cup \{0\}$ actually forms a ring and is called the valuation ring of ν. We will look at the specific example when $K = \mathbb{Q}$.

Fix a prime number p. For any $r = \dfrac{m}{n} \in \mathbb{Q}^\times$, note that it can be written as

$$r = \frac{m}{n} = p^k \frac{m'}{n'} \quad \text{for some integer } k,\ \ p \nmid m',\ \ p \nmid n'$$

(We are essentially taking out the prime factor p, if any, from both the numerator and the denominator. Note that k can be negative or 0.) Define $\nu_p(r) = k$.

(a) Show that ν_p is a discrete valuation on $\mathbb{Q}$.

(b) What is the valuation ring R_{ν_p} ? Identify the units in R_{ν_p}.

(c) Show that $M_{\nu_p} := \{x \in R_{\nu_p} \mid \nu_p(x) > 0\}$ is an ideal of R_{ν_p} and the quotient R_{ν_p}/M_{ν_p} is a field.

Remark. ν_p is called the p-adic valuation on $\mathbb{Q}$. This idea is sort of the starting point of what we call p-adic numbers. These numbers are fascinating and widely studied in the fields of Algebraic Number Theory and Algebraic geometry. We may have some more future bonus problems exploring this idea.