MAT401 **Due Date : Thursday, July 7, 11:59 pm**
SUMMER 2022 **Submission : Crowdmark**

Problem Set 4

As in lectures, by rings we shall mean a commutative ring with identity, unless otherwise specified. Textbook questions are from "Galois Theory" (2nd edition) by Joseph Rotman.

1. (a) Textbook exercise 56.
   (b) Textbook exercise 59.
   (*This result is going to be very important and useful in the future.*)

2. Consider $p(x) = x^3 + 2x + 1 \in \mathbb{Z}_3[x]$ from midterm. You showed $p(x)$ is irreducible and so $E := \mathbb{Z}_3[x]/(p(x))$ is a field. We saw in lectures, $p(x)$ has a root in E namely : $\bar{x} = x + (p(x))$ and every element in E can be written as

$$\overline{ax^2 + bx + c} = ax^2 + bx + c + (p(x)) \quad \text{for some } \; a, b, c \in \mathbb{Z}_3$$

   (a) What is $\overline{x^2 + 2} \cdot \overline{x^2 + x + 2}$ in E ? (i.e. reduce it to the above form)
   (b) Let us consider the polynomial p over E, i.e. $p(y) = y^3 + \bar{2}y + \bar{1} \in E[y]$. Then we can factorize $p(y) = (y - \bar{x})q(y)$ in $E[y]$. Find $q(y)$.

3. Let R be a ring. Prove that if every proper ideal of R is prime, then R is a field. (Note : You proved the easier finite version in the midterm.)

4. Let R be a ring and I be an ideal of R. The **radical** of I is defined as :

$$\sqrt{I} := \{r \in R \; : \; r^n \in I \; \text{ for some } n \in \mathbb{Z}_{>0}\}$$

   (a) Show that $\sqrt{I}$ is an ideal of R containing I.
   (b) Show that if I is a maximal ideal, then $\sqrt{I} = I$.
   (c) Consider the ring $\mathbb{Q}[x]$ and let $I = (x^4) \subseteq \mathbb{Q}[x]$. What is $\sqrt{I}$?

5. Let R be a ring.

   (a) Show that the radical of the zero ideal is contained in the intersection of all prime ideals of R, i.e.

$$\sqrt{(0)} \;\subseteq \bigcap_{\mathfrak{p} \subseteq R \text{ ,prime}} \mathfrak{p}$$

   (*Note : This is in fact an equality, but the other way is somewhat harder to prove.*)

   (b) $\sqrt{(0)}$ is often called the **nilradical** of R and elements of $\sqrt{(0)}$ are called **nilpotent** elements. A ring is called **reduced** if $\sqrt{(0)} = (0)$.
   Show that the quotient ring : $\dfrac{R}{\sqrt{(0)}}$ is a reduced ring.