Problem Set 6

Textbook questions are from "Galois Theory" (2nd edition) by Joseph Rotman.

1. Textbook exercise 72.
2. Textbook exercise 77.
3. (a) Find the irreducible polynomial $p(x)$ of $\sqrt{2}+i$ over $\mathbb{Q}$.
   (b) Find the splitting field of $p(x)$ over $\mathbb{Q}$.
   (c) What is the degree of this splitting field over $\mathbb{Q}$? Justify.
4. Let $K = \mathbb{Q}(\alpha_1, \cdots, \alpha_n)$ where $\alpha_i^3 \in \mathbb{Q}$ for all $i = 1, 2, \cdots, n$.
   (a) Determine all possible prime factors of $[K : \mathbb{Q}]$. Justify your answer.
   (b) Conclude that $\sqrt[5]{2} \notin K$.
5. Let F be any field and let $f(x) = a_n x^n + \cdots + a_0 \in F[x]$. For any matrix $A_{t\times t}$, with entries in F, one can define $f(A) = a_n A^n + \cdots + a_0 I_{t\times t}$, $I_{t\times t}$ being the identity matrix. Consider $I_A := \{p(x) \in F[x] \mid p(A) = 0\}$. Show that I_A is an ideal, generated by a monic polynomial of smallest degree. This, in linear algebra, is called the *minimal polynomial* of a matrix.

   Note : If needed, you may use the Cayley-Hamilton Theorem, which says every square matrix satisfies its own characteristic polynomial.

6. Let E be a finite extension of F and let $\alpha \in E$. We know E can be treated as a vector space over F. Consider the map $T_\alpha : E \to E$ given by $T_\alpha(x) = \alpha x$
   (a) Show that T_α is a linear transformation from E to E (as vector space over F).
   (b) Choose a basis for E and let A_α be the matrix of T_α with respect to this chosen basis. Show that for any $f(x) \in F[x]$, $f(\alpha) = 0$ iff $f(A_\alpha) = 0$.
   (c) Show that the irreducible polynomial of α over F is same as the minimal polynomial of A_α over F.