Problem Set 7

Textbook questions are from "Galois Theory" (2nd edition) by Joseph Rotman.

1. Textbook exercise 84.

2. Textbook exercise 85.
Remark : Once you have solved these two exercises, I will recommend you to go through the proof of Lemma 73 one more time to better understand the result.

3. For the following polynomials in $\mathbb{Q}[x]$, determine the Galois group of their splitting fields over $\mathbb{Q}$. Justify your answers.
   (a) $f(x) = (x^2-3)(x^2-5)(x^2-7)$
   (b) $g(x) = \prod_{i=1}^{n}(x^2-p_i)$, where $p_1, p_2, \cdots, p_n$ are distinct primes.
   (c) $h(x) = (x^2+3)(x^2-5)$

4. Let $f(x),\ g(x) \in \mathbb{Q}[x]$ be solvable by radicals. Show that fg is solvable by radicals.
Hint : Let E, F, F' be the splitting fields of $fg,\ f,\ g,$ over $\mathbb{Q}$, respectively. So F, F' are subfields of E. Show that $Gal(E/\mathbb{Q})$ is isomorphic to a subgroup of $Gal(F/\mathbb{Q}) \times Gal(F'/\mathbb{Q})$, i.e. there is an injective homomorphism $\phi : Gal(E/\mathbb{Q}) \to Gal(F/\mathbb{Q}) \times Gal(F'/\mathbb{Q})$.

5. Let $\alpha \in \mathbb{C}$ be constructible .
   (a) Show that α is algebraic over $\mathbb{Q}$.
   (b) If $p_\alpha(x)$ is the minimal polynomial of α over $\mathbb{Q}$, what can you say about $\partial(p_\alpha)$? Use this to construct an algebraic number that is not constructible.
   (c) Show that the angle 72^o can be constructed using only a straightedge and a compass.
   Hint : Show that θ can be constructed as an angle if $\cos\theta$ is constructible. You may use $\cos 72^o = \frac{\sqrt{5}-1}{4}$ to then show that it is constructible.

2% **Bonus Problem.** This problem is based on the following interesting question asked by a student during the lectures : Are subfields of a radical extension, radical ? The answer is no, and this problem is about producing an example.

(a) Let $\zeta_7 = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$ be a 7th root of unity. So $\mathbb{Q}(\zeta_7)$ is a radical extension. Consider the subfield $L = \mathbb{Q}(\zeta_7 + \zeta_7^{-1}) \subseteq \mathbb{Q}(\zeta_7)$. Show that $[L : \mathbb{Q}] = 3$.

(b) So there are no proper subfields between, $\mathbb{Q} \subseteq L$. If L was radical, it must be a pure extension, i.e. $L = \mathbb{Q}(\alpha)$ with $\alpha^n \in \mathbb{Q}$ for some n. Show that $n = 3$.
Hint : Minimal polynomial of α is of degree 3. Figure out what its roots look like. Note $L \subseteq \mathbb{R}$, and only real roots of unity are ± 1.

(c) Derive a contradiction by showing that the minimal polynomial of α must have repeated roots then.