L-FUNCTIONS OF THE PICARD FAMILY OF CURVES

SAMPRIT GHOSH

ABSTRACT. Let C be a smooth projective algebraic curve over $\mathbb{Q}$. Then the L-function of C is conjectured to have analytic continuation on the entire complex plane. Deligne conjectured that the value of the L-function evaluated at critical points is an algebraic number times the determinant of the real period matrix associated with C. The periods of such curves are also expressible in terms of hypergeometric functions.
In this paper we initially look at Picard family of genus 3 curves. We verify Deligne's conjecture for certain genus 2 and genus 3 curves. We also explore period matrices expressible as hypergeometric functions. Eventually we examine Gaussian hypergeometric functions and give a finite field analogue of Appel-Lauricella hypergeometric functions.

1. INTRODUCTION

Let C be a smooth projective curve of genus g defined over a number field K. For our purposes we restrict ourself to $\mathbb{Q}$. The L-function of C is an analytic function of one complex variable s defined as an Euler product

$$L(C,s) := \prod_{\mathfrak{p}} L_{\mathfrak{p}}(C,s)$$

where $\mathfrak{p}$ is taken over all primes in K. The local L-factor is of the form

$$L_{\mathfrak{p}}(C,s) := \frac{1}{f((N\mathfrak{p})^{-s})}$$

where $N\mathfrak{p}$ is the norm of $\mathfrak{p}$ and $f(t) \in Z[t]$ is a polynomial with integer coefficients depending on $\mathfrak{p}$. An invariant associated with C is the conductor of the L-function. It is a positive real number of the form

$$N := \prod_{\mathfrak{p}} N\mathfrak{p}^{f_{\mathfrak{p}}}$$

Here $f_{\mathfrak{p}}$, the exponent at $\mathfrak{p}$ is a non-negative integer which is zero for all but finitely many $\mathfrak{p}$.

Some nice examples are (for K=$\mathbb{Q}$)

The **Riemann-Zeta function** : $(N = 1)$

$$\zeta(s) = \prod_{p}(1-p^{-s})^{-1} = \sum_{n=1}^{\infty} n^{-s}$$

The **Dirichlet L-function**: (N : a divisor of k)

$$L(\chi,s) = \prod_{p}(1-\chi(p)p^{-s})^{-1} = \sum_{n=1}^{\infty} \chi(n)n^{-s}$$

where χ is a Dirichlet Character (modulo k say)i.e. a group homomorphism from $(\mathbb{Z}/k\mathbb{Z})^\star$ to the non-zero Complex numbers, roots of unity to be specific.
The study of L-functions is still an area with many conjectures. For example, it is conjectured that $L(C,s)$ has analytic continuation to the entire complex plane and has a functional equation of the form

$$\Lambda(C,s) = \pm\Lambda(C, 2-s),$$

where

$$\Lambda(C,s) = N^{s/2}(2\pi)^{-gs}\Gamma(s)^g L(C,s)$$

Also there are conjectures on values of an L-function at critical points. e.g. Deligne's Conjecture, Birch and Swinnerton-Dyer Conjecture. One of our aim of this project is to computationally verify some of these conjectures for certain family of curves. We first consider the Picard family.
The Picard Family of curves is given by :

$$C := y^3 = x(1-x)(1-\lambda x)(1-\mu x)$$

This is a two parameter family. Calculating the genus of the curve from the Riemann- Hurwitz formula

$$2 - 2g = 2n - \sum e$$

where e is the ramification number, $e =$(number of sheets coming together - 1)
so

$$2 - 2g = 2 * 3 - 10$$

$$\implies g = 3$$

This is a genus 3 curve. Therefore for any prime p, the Zeta function over the finite field $\mathbb{F}_p$ is known to be of the form

$$\mathbb{Z}(C/\mathbb{F}_p, t) = exp(\sum_{\nu=1}^{\infty} N_\nu \frac{t^\nu}{\nu}) \tag{1}$$

where N_ν is the number of points on the curve over the finite field $\mathbb{F}_{p^\nu}$ including the point at infinity. Now from Dwork's theorem on rationality of zeta functions (1) is known to be a rational function. Moreover for 'good' primes p,(C having good reduction at p) i.e. primes not dividing the conductor N it is known to be of the following form :

$$\mathbb{Z}(C/\mathbb{F}_p, t) = \frac{1 + at + bt^2 + ct^3 + pbt^4 + p^2at^5 + p^3t^6}{(1-t)(1-pt)} \tag{2}$$

Thus local L_p factor is given by

$$L_p(C,s) = \frac{1}{1 + ap^{-s} + bp^{-2s} + cp^{-3s} + pbp^{-4s} + p^2ap^{-5s} + p^3p^{-6s}} \tag{3}$$

Thus to compute the L-function we wish to compute the number of points on the curve C over $\mathbb{F}_q$ and thereby compute the coefficients a, b, c.

2. Character Sums

The first approach taken to the study of point counting on curves over finite fields is to express the number of points in terms of character sums.

Definition 2.1. *Let G be a finite Abelian group (more generally, locally compact Abelian group). A* ***character*** *is a homomorphism*

$$\chi : G \to \mathbb{C}_1^* = \{z \in \mathbb{C} : |z| = 1\}$$

Note that the set of characters denoted by $\hat{G}$ is also an abelian group.

Example 2.2. $G = \mathbb{Z}/n$. Let $\zeta_n \in \mathbb{C}_1^*$ be a primitive n-th root of unity. For instance $\zeta_n = e^{\frac{2\pi i}{n}}$. Fix an integer $j \in \mathbb{Z}/n$ The homomorphism $\chi_j : \mathbb{Z}/n \to \mathbb{C}_1^*$ by $\chi_j(a) = \zeta_n^{ia}$ defines a character. In fact, $j \mapsto \chi_j : \mathbb{Z}/n \to \widehat{\mathbb{Z}/n}$ is an isomorphism.

Let $\mathbb{F}_q$ be a finite field where $q = p^a$ for some prime p. The $\mathbb{F}_q/\mathbb{F}_p$ is a Galois extension and $Gal(\mathbb{F}_q/\mathbb{F}_p) \sim C_a =< 1, \phi, \phi^2, ..., \phi^{a-1} >$ where ϕ is the Frobenius map. There are two types of characters of the field $\mathbb{F}_q$ for the two groups namely additive and the multiplicative group. The multiplicative group $(\mathbb{F}_q^*, \times) \simeq C_{q-1} = \{1, \alpha, \alpha^2, ..., \alpha^{q-1}\}$. So the set of characters of this group, called the multiplicative characters, are given by $\widehat{\mathbb{F}_q^*} = \{\chi_i : i \in \mathbb{Z}/q-1\}$,where $\chi_i(z) = \zeta_{q-1}^{ilog_\alpha(z)}$.

One may very easily prove that for any finite abelian group G and any function $f : G \to \mathbb{C}$ can be uniquely expressed as a linear combination of characters:

$$f(a) = \sum_{\chi \in \widehat{G}} \hat{f}(\chi)\chi(a)$$

where the function $\hat{f} : \widehat{G} \to \mathbb{C}$ is caled the **Fourier transform** and is given by:

$$\hat{f}(\chi) = \frac{1}{|G|} \sum_{g \in G} f(g)\overline{\chi(g)}$$

Using the above fact one can prove the following:

Proposition 2.3. *Let $a \in \mathbb{F}_q^*$ and fix an integer $N \geq 1$. Define : $f(a) := \#\{x \in \mathbb{F}_q^\star : x^N = a\}$. $\{$For simplicity we may assume that $N \mid (q-1)\}$ then$f(a) = \sum_{\chi^N=1} \chi(a)$ i.e. sum over all the characters of $\mathbb{F}_q^*$ such that $\chi^N = 1$.*

Proof. **Result :** Suppose $\chi : G \to \mathbb{C}_1^*$ is a character and $\chi \neq Id$ then $\sum_{g \in G} \chi(g) = 0$.

Since $\chi \neq Id$, $\exists g_0 \in G$ such that $\chi(g_0) \neq 1$ and therefore as g runs over all elements of G, gg_0 also runs over all elements of G. Thus

$$S = \sum_{g \in G} \chi(g) = \sum_{g \in G} \chi(gg_0) = \chi(g_0) \sum_{g \in G} \chi(g) = \chi(g_0)S \implies S = 0.$$

as $N \mid q-1$ we have an exact sequence,

$$1 \to \mu_N \to \mathbb{F}_q^* \to (\mathbb{F}_q^*)^N \to 1$$

where μ_N is the cyclic group of order N. $(\mathbb{F}_q^*)^N$ is cyclic of order $\frac{q-1}{N}$. Again by Fourier analysis, we have : $f(a) = \sum_{\chi \in \widehat{G}} \hat{f}(\chi)\chi(a)$ and $\hat{f}(\chi) = \frac{1}{q-1}\sum_{b\in G} f(b)\overline{\chi(b)}$

By the above exact sequence, $f(b) = \begin{cases} 0 & \text{if } b \notin (\mathbb{F}_q^*)^N \\ N & \text{if } b \in (\mathbb{F}_q^*)^N \end{cases}$

$$\text{So, } \hat{f}(\chi) = \frac{N}{q-1} \sum_{b\in(\mathbb{F}_q^*)^N} \overline{\chi(b)}$$

But from the result above, the sum is 0 if $\chi \mid_{(\mathbb{F}_q^*)^N} \neq Id$.

And if $\chi \mid_{(\mathbb{F}_q^*)^N} = Id \iff \chi^N = 1$ and the sum becomes $\frac{N}{q-1} \cdot \#(\mathbb{F}_q^*)^N = \frac{N}{q-1}\frac{q-1}{N} = 1$

Therefore $\hat{f}(\chi) = \begin{cases} 0 & \text{if } \chi^N \neq Id \\ 1 & \text{if } \chi^N = Id \end{cases}$

$$\text{So, } f(a) = \sum_{\chi} \hat{f}(\chi)\chi(a) = \sum_{\chi^N=1} \chi(a)$$

□

Thus we see that character sums are an important tool for point counting. We now state some more facts which will help us further to compute the number of points and hence the local L-factors for the Picard family of curves. Consider a more general family, $X : y^n = f(x)$. One may consider it as a covering of the projective line $\mathbb{P}_x^1$ by the canonical projection mapping $(x, y) \mapsto x$. Then the zeta function can be factored as follows :

$$Z(X/\mathbb{F}_p, t) = Z(\mathbb{P}^1/\mathbb{F}_p, t) \cdot L(X/\mathbb{P}^1, t)$$

The function field $\mathbb{F}_p(x, y)$ of the curve X is a Galois extension over $\mathbb{F}_p(x)$, the function field of $\mathbb{P}_x^1$. If $p \equiv 1(\text{mod } n)$ then the ground field contains n^{th} roots of unity and thus the automorphisms are given by $(x, y) \mapsto (x, \zeta_n y)$. And the L-factor in the zeta function can further be written as :

$$L(X/\mathbb{P}_x^1, t) = \prod_{\substack{\chi:\mu_n \to \mathbb{C}_1^* \\ \chi \neq Id}} L(\mathbb{P}_x^1, \chi)$$

The N_ν in a zeta function as in (1) can be written as $N_\nu = \sum_{t\in X(\mathbb{F}_{p^\nu})}$, let us define $S_\nu(\chi) = \sum_{z\in X(\mathbb{F}_{p^\nu})}$. The it is known that

$$L(\mathbb{P}^1, \chi) = exp(\sum_{\nu=1}^{\infty} S_\nu(\chi)\frac{t^\nu}{\nu})$$

This is in fact a polynomial in t. Another way is to look at it as the characteristic equation of a Frobenius map acting on the cohomology $H^1_{ét}(X \otimes \overline{\mathbb{Q}}, \mathbb{Q}_l)$ i.e.

$$det(1 - t\rho(Frob_p) \mid H^1_{ét}(X \otimes \overline{\mathbb{Q}}, \mathbb{Q}_l)) \text{ where } \rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(H^1_{ét}(X \otimes \overline{\mathbb{Q}}, \mathbb{Q}_l))$$

3. Computations using Character sums

We have written codes in sage to implement character sums to compute the coefficients for the local L-factors for Picard family of curves. Now observe that since it is a genus 3 curve therefore we only have to compute coefficients a, b, c. Thus we need only consider characters in $\mathbb{F}_{p^\nu}$, for $\nu = 1, 2, 3$. Here is the code to do so:

```
K = CyclotomicField(3);

def ptshyper(p, q,r):

   F = GF(p)
   F1 = GF(p^2, 'a')
   F2 = GF(p^3, 'c')


   def chi(x):
      z = K.gen()
      b = F.multiplicative_generator()
      if x==0:
         return 0
      return z^(x.log(b))

   def chi1(x):
      z = K.gen()
      a = F1.multiplicative_generator()
      if x==0:
         return 0
      return z^(x.log(a))

   def chi2(x):
      z = K.gen()
      c = F2.multiplicative_generator()
      if x==0:
         return 0
      return z^(x.log(c))

   if p%3 == 1:
      return [p, sum([chi(x*(1-x)*(1-q*x)*(1-r*x)) for i,x in
         enumerate(F)]) + sum([chi(x*(1-x)*(1-q*x)*(1-r*x))^2 for i,x
         in enumerate(F)]),sum([chi1(x*(1-x)*(1-q*x)*(1-r*x)) for i,x
         in enumerate(F1)]) + sum([chi1(x*(1-x)*(1-q*x)*(1-r*x))^2 for
         i,x in enumerate(F1)]),sum([chi2(x*(1-x)*(1-q*x)*(1-r*x)) for
         i,x in enumerate(F2)]) + sum([chi2(x*(1-x)*(1-q*x)*(1-r*x))^2
         for i,x in enumerate(F2)])]

   if p%3 == 2:
      return [p,0,sum([chi1(x*(1-x)*(1-q*x)*(1-r*x)) for i,x in
         enumerate(F1)]) + sum([chi1(x*(1-x)*(1-q*x)*(1-r*x))^2 for
         i,x in enumerate(F1)]),0]
```

This function in the code takes input the prime p, λ as 'q' and μ as 'r' and returns N_1, N_2, N_3.Observe that if $p \equiv 2 (\bmod 3)$ then there is no character of order 3 in $\mathbb{F}_{p^\nu}$ for odd ν. The next function calls this 'ptshyper' function and finally returns the prime along with the coefficients a, b, c.

```
def coefsl(p,q,r):
      PointList = ptshyper(p,q,r)
      N1 = PointList[1]
      N2 = PointList[2]
      N3 = PointList[3]
      return [p, N1, (N1^2+N2)/2, ((N1)^3+(3*N1*N2)+(2*N3))/6]
```

Example 3.1. Here we consider the Picard family with parameters $\lambda = 2, \mu = 3$,

$$C := y^3 = x(1-x)(1-2x)(1-3x)$$

2 and 3 are probably going to be 'bad primes', so we compute from 5.

```
for n in range(5, 42):
   if is_prime(n):
     print n, coefsl(n, 2, 3)
```

The following output was given :

```
5 [5, 0, -3, 0]
7 [7, -3, -6, 43]
11 [11, 0, 6, 0]
13 [13, -3, 12, -47]
17 [17, 0, -3, 0]
19 [19, 6, -15, -200]
23 [23, 0, 42, 0]
29 [29, 0, 33, 0]
31 [31, 3, -12, 25]
37 [37, 6, 30, 358]
41 [41, 0, -3, 0]
```

To check correctness we also have tested the coefficients for Weil number. For exaample, for the prime 13 :

```
R.<t> = PolynomialRing(Complexes()); R
f = 1+(-3)*t+(12)*t^2+(-47)*t^3+13*(12)*t^4+13^2*(-3)*t^5+13^3*t^6; f

2197.00000000000*t^6 - 507.000000000000*t^5 + 156.000000000000*t^4 -
   47.0000000000000*t^3 + 12.0000000000000*t^2 - 3.00000000000000*t +
   1.00000000000000
```

```
rts = f.roots(); rts

[(-0.173471517687619 - 0.216404042185517*I, 1), (-0.173471517687619 +
   0.216404042185517*I, 1), (0.0410260210616175 - 0.274299002037792*I,
   1), (0.0410260210616175 + 0.274299002037792*I, 1),
   (0.247830112010617 - 0.124512298604924*I, 1), (0.247830112010617 +
   0.124512298604924*I, 1)]
68


[CC(sqrt(13))*rts[i][0].abs() for i in [0..3] ]

[1.00000000000000, 1.00000000000000, 1.00000000000000,
   1.00000000000000]
```

But it seemed that working with character sums is very computationally expensive. even for the prime 43 it takes a very long time. And in order to compute or at least get an idea of the global L-function we need to compute the local L-factors for decently large number of primes.So we tried to come up with something different, and magma seems to provide a built-in zeta function calculator.

```
R<t> := PolynomialRing(Rationals());
K<z> := CyclotomicField(3);
S<t> := PolynomialRing(K);

zet := function(n)
      P<x,y,z> := ProjectiveSpace(GF(n),2);
      C := Curve(P,y^3*z -x*(z-x)*(z-2*x)*(z-3*x));
      return [n, Coefficient(Numerator(ZetaFunction(C, 1)), 1),
         Coefficient(Numerator(ZetaFunction(C, 1)), 2),
         Coefficient(Numerator(ZetaFunction(C, 1)), 3)];
      end function;
```

For the same example we have written codes and the output is given in Appendix A.

It's quite much faster for small primes but again for primes larger than 200 it takes much longer. Our initial plan was to compute enough number of local L-factors and then put them in the 'lfun' package in pari gp which will in turn compute the global L-function. But due to inability to compute enough data and technological constraints of the package we couldn't. Even computing the conductor for curves of this family is challenging.

In fact computing conductors in general is non-trivial. Though one can easily compute the conductor for elliptic curves from the famous "Tate's Algorithm". There is also an algorithm available for genus 2 hyperelliptic curves given by Qing Liu, it doesn't compute the exponent of the prime 2 in the conductor though. Therefore we decided to start looking at more simpler families like hyperelliptic family of curves and curves of Complex multiplication.

It so happens that magma has a built-in function to generate L-functions for Hyperelliptic curves.

Example 3.2. Consider the hyperelliptic curve $C := y^2 = x^6 - 14x^5 + 61x^4 - 106x^3 + 66x^2 - 8x - 3$, This is a genus 2 curve. We computed its conductor to be 961. We can simply compute the L-function of this curve in magma by 'LSeries' function. We evaluated the value of the L-function at 1, which came out to be: **0.449287723876040786113329668188**

We conclude this section by looking at 'bad primes' and what may the form of the local factors look like for such primes for simpler curves.

Let E be an elliptic curve over $\mathbb{Q}$ with Weierstrass form i.e.

$$f(x,y) = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6$$

which has a singular point at $P(x_0, y_0)$ say. One can write the Taylor series of f at (x_0, y_0) as follows:

$$f(x,y) - f(x_0,y_0) = \lambda_1(x-x_0)^2 + \lambda_2(x-x_0)(y-y_0) + \lambda_3(y-y_0)^2 - (x-x_0)^3 \text{ where } \lambda_i \in \mathbb{Q}$$

(which can be factored as)

$$= [(y-y_0) - \alpha(x-x_0)][(y-y_0) - \beta(x-x_0)] - (x-x_0)^3 \text{ where } \alpha, \beta \in \overline{\mathbb{Q}}$$

P is a node or a cusp according as $\alpha \neq \beta$ or $\alpha = \beta$. Now let us assume that $E/\mathbb{Q}$ has a minimal model with same Weierstrass form with coefficients in $\mathbb{Z}$. Let p be a prime. By reducing each coefficient a_i modulo p we obtain a cubic curve $\tilde{E}$ over $\mathbb{F}_p$. If $\tilde{E}$ is non-singular then we say that E has good reduction at p. Otherwise it has bad reduction at p. If $\tilde{E}$ has a cusp, E has additive reduction at p. If $\tilde{E}$ has a node then E has a multiplicative reduction at p. If the slopes of the tangent lines are in $\mathbb{F}_p$ then the reduction is said to be split multiplicative.

Fact : $E/\mathbb{Q}$ with coefficients in $\mathbb{Z}$, If E has bad reduction at p then $p|\Delta$ where Δ is the discriminant of the curve.
We can apply the same theory to hyperelliptic curve as well. Consider the following hyperelliptic curve:

$$C := y^2 = x(1-x)(1-2x)(1-3x)(1-5x)$$
$$\implies y^2 = 30x^5 - 61x^4 + 41x^3 - 11x^2 + x$$

the only primes dividing the conductor are 2,3,5. The form of the local $L-$factors for the primes except 2 can be guessed as follows:
For the prime 5, the equation becomes : $y^2 = -x^4 + x^3 - x^2 + x$, Normalization leads to a genus 1 elliptic curve $E := y^2 = x^3 + x^2 + x + 1$. Therefore

$$L(C/\mathbb{F}_5, t) = L(E/\mathbb{F}_5) \cdot L(\text{“}torus\text{”}, t)$$

Now $L(E/\mathbb{F}_5) = 1 - at + 5t^2$ where $a = 5 + 1 - \#E(\mathbb{F}_5) = 6 - 8 = -2$ and $L(\text{“}torus\text{”}, t)$ is $(1+t)$ or $(1-t)$
For the prime 3, the equation becomes : $y^2 = -x^4 - x^3 + x^2 + x$, Normalization leads to $y^2 = x - x^2$ which is a genus zero curve. So

$$L(C/\mathbb{F}_3, t) = L(\text{“}torus\text{”}, t)$$

which in this case can be $(1+t)^2$, $(1-t)^2$, $(1+t^2)$, $(1-t^2)$

4. Period matrices

Consider any algebraic curve $X/\mathbb{Q}$ of genus g. Then

$$dim\ H^0(X, \Omega^1_{X/\mathbb{Q}}) = g \text{ and } dim\ H_1(X, \mathbb{Z}) = 2g$$

Therefore $\exists\ g$ holomorphic 1-forms $\omega_1, \omega_2, ..., \omega_g$ defined over $\mathbb{Q}$ which are linearly independent. i.e. they form a basis of the space $H^0(X, \Omega^1_{X/\mathbb{Q}})$. One can determine these differential 1-forms by using the technique of "divisors" from algebraic geometry. Let $\{\alpha_1, \alpha_2, ..., \alpha_g, \beta_1, \beta_2, ..., \beta_g\}$ be a simplectic basis of $H_1(X, \mathbb{Z})$. Then

$$M_{g\times 2g} = \begin{bmatrix} \int_{\alpha_1}\omega_1 & \cdot & \cdot & \cdot & \int_{\alpha_g}\omega_1 & \int_{\beta_1}\omega_1 & \cdot & \cdot & \cdot & \int_{\beta_g}\omega_1 \\ \int_{\alpha_1}\omega_2 & \cdot & \cdot & \cdot & \int_{\alpha_g}\omega_2 & \int_{\beta_1}\omega_2 & \cdot & \cdot & \cdot & \int_{\beta_g}\omega_2 \\ \cdot & & & & & & & & & \cdot \\ \cdot & & & \cdot & & \cdot & & \cdot & & \cdot \\ \cdot & & & & & & & & & \cdot \\ \int_{\alpha_1}\omega_g & \cdot & \cdot & \cdot & \int_{\alpha_g}\omega_g & \int_{\beta_1}\omega_g & \cdot & \cdot & \cdot & \int_{\beta_g}\omega_g \end{bmatrix}$$

M is called the period matrix. These integrals in the matrix M can be expressed in terms of Hypergeometric functions.
Consider a curve $X := f(x, y) = 0$ over $\mathbb{Q}$ and the complex conjugation map $(x, y) \mapsto (\overline{x}, \overline{y})$. Since coefficients of X are in $\mathbb{Q}$ if $f(x, y) = 0$ then $f(\overline{x}, \overline{y}) = 0$ i.e. $(\overline{x}, \overline{y})$ is also in the variety. This is a C^∞-function. Let us call it F_∞ and consider

$$F_\infty : H^1(X, \mathbb{Q}) \to H^1(X, \mathbb{Q})$$

The matrix of this transformation is :

$$\begin{pmatrix} 0 & I_g \\ I_g & 0 \end{pmatrix} \sim \begin{pmatrix} I_g & 0 \\ 0 & -I_g \end{pmatrix}$$

So, $H^1(X, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$

Deligne's Conjecture : Let M be a "motive" and $L(M, s)$ be the motivic L-function. Then there exist two complex numbers $c_\pm(M)$ such that for any "critical point" m we have

$$\frac{L(M, m)}{c_\pm(M)} \in \overline{\mathbb{Q}}$$

Note : Recall the local L-factors can be realised as the characteristic equations of certain Frobenius map acting on certain *étale* cohomology. A motive $M/\mathbb{Q}$ of dimension n gives rise to a Galois representation

$$\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(H^1_{\acute{e}t}(M \otimes \overline{\mathbb{Q}}, \mathbb{Q}_l)) \ = GL_n(\mathbb{Q}_l)$$

and the motivic L-function is given by

$$L(M, s) = \prod_p \frac{1}{det(1 - p^{-s}\rho(Frob)_p | H^1_{\acute{e}t}(M \otimes \overline{\mathbb{Q}}, \mathbb{Q}_l))}$$

with modifications at finitely many primes p where M has bad reduction.
Now for algebraic curves, the complex number $c_\pm$ can be computed as the determinant of matrices with real periods as entries. For a genus g curve we can take g linearly independent elements from the space $H_1(X, \mathbb{Z})$ which are fixed under the transformation F_∞, and integrating the 1-forms over

these cycles will give real periods. Thus we can get a $g \times g$ matrix with real entries. Moreover in such cases the ratio of the special value of the L-function and $c_{\pm}$ (which is now just the determinant of the $g \times g$ matrix) is infact conjectured to be a rational number.
Using magma we computed some period matrices and special values of L-functions given below.

Example 4.1. Consider the curve as in example **3.2** where we already evaluated the L-function at 1. Here we give the code to compute the period matrix.

```
C<I> := ComplexField(50);
R<x> := PolynomialRing(C);

f:= x^6-14*x^5+61*x^4-106*x^3+66*x^2-8*x-3; f

A := AnalyticJacobian(f) ; A

M:= BigPeriodMatrix(A); M

X := Matrix(C, 2, 4, [Real(M[i][j]) : j in [1..4], i in [1..2]]); X
Y := Matrix(C, 2, 4, [Im(M[i][j]) : j in [1..4], i in [1..2]]); Y

S := BlockMatrix(2,1, [X, Y]);S
T := BlockMatrix(2,1, [X, -Y]);T

F := S^-1*T;
expr := F^2; expr

seq := [1: i in [1..4]]; seq
Id := DiagonalMatrix(C, 4, seq); Id

Z := F+Id; Z

K := Matrix(C, 4, 4, [Round(Z[i][j])/2 : j in [1..4], i in [1..4]]); K1

N :=M*K; N
list := [Real(Ntr[1][1]), Real(Ntr[1][2]), Real(Ntr[3][1]),
   Real(Ntr[3][2])]; list
W := Matrix(C, 2, 2, list); W

Delta:=Determinant(W); Delta
```

(outputs of the above commands are given in the Appendix - B)

Magma computes something called a big period matrix (since the curve has genus 2, it's a 4×2 matrix) for us, from which we compute 'N'. Then from 'N' we choose 2 linearly independent columns to make 'W', and finally compute its determinant. Which in this case was :

$$\mathbf{3.369657929070305895849972511409429350058351769092 8}$$

and finally the ratio of $L(C, 1)$ as computed in example **3.2** and the above determinant was:

$$\mathbf{0.13333333333333333333333333333} \sim \frac{2}{15}$$

So it satisfies the Deligne's conjecture.

Example 4.2.

$$C := y^2 = x^8 - 4x^7 - 6x^6 - 4x^5 - 9x^4 + 4x^3 - 6x^2 + 4x + 1$$

Interestingly this is a genus 3 curve. We computed its period matrix and the value of its L-function using magma and the ratio was given to be :

$$\textbf{-0.562499999999999999999999999999} \sim -\frac{9}{16}$$

In the next example we compute the period matrix in a different and more theoretical way. But before that let us introduce hypergeometric functions.

4.1. **Hypergeometric functions.** The hypergeometric function in one parameter is given by :

$${}_pF_q\left(\begin{array}{cccc} a_1 & a_2 & \cdots & a_p \\ & b_1 & \cdots & b_q \end{array} \middle| z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\frac{x^k}{k!}$$

where $(a)_k$ denotes the Pochhammer symbol given by $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\cdots(a+k-1)$ This sort of hypergeometric functions arises as solutions to certain differential equations. Also some specific type of integrals can also be expressed in terms of these function. Here are some examples :
${}_{p+1}F_p\left(\begin{array}{cccc} a_1 & a_2 & \cdots & a_p+1 \\ & b_1 & \cdots & b_p \end{array} \middle| z\right)$ is a solution to the differential equation :

$$[\theta(\theta+b_1-1)\cdots(\theta+b_p-1) - z(\theta+a_1)(\theta+a_2)\cdots(\theta+a_{p+1})]y = 0 \text{ where } \theta = z\frac{d}{dz}$$

Also there are integrals that can be expressed in terms of Hypergeometric functions.
Euler's integral formula: when Re(b_1)>Re(a_2)> 0

$${}_2F_1\left(\begin{array}{cc} a_1 & a_2 \\ & b_1 \end{array} \middle| z\right) = \frac{\Gamma(b_1)}{\Gamma(a_2)\Gamma(b_1-a_2)}\int_0^1 x^{a_2-1}(1-x)^{b_1-a_2-1}(1-zx)^{-a_1}dx \tag{4}$$

More generally,

$${}_{r+1}F_r\left(\begin{array}{cccc} a_1 & a_2 & \cdots & a_r+1 \\ & b_1 & \cdots & b_r \end{array} \middle| \lambda\right) =$$

$$\frac{\Gamma(b_r)}{\Gamma(a_{r+1})\Gamma(b_r-a_{r+1})}\int_0^1 x^{a_{r+1}-1}(1-x)^{b_r-a_{r+1}-1}{}_rF_{r-1}\left(\begin{array}{cccc} a_1 & a_2 & \cdots & a_r \\ & b_1 & \cdots & b_{r-1} \end{array} \middle| \lambda x\right)dx \tag{5}$$

Example 4.3. Consider the Legendre family of elliptic curves:

$$E_\lambda/\mathbb{Q} := y^2 = x(1-x)(1-\lambda x),\ \lambda \neq 0, 1$$

This is a genus 1 curve. Thus if α, β be two basis elements then, $H_1(E_\lambda, \mathbb{Z}) = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$. Using the concept of "divisors" we can find a differential form $\omega = \frac{dx}{y}$ which forms a basis of $H^0(E_\lambda, \Omega^1_{E_\lambda/\mathbb{Q}})$. So that the period matrix is given by

$$\left(\begin{array}{cc} \int_\alpha \omega & \int_\beta \omega \end{array}\right)$$

Then $\int_\alpha \omega$, $\int_\beta \omega$ are linear combination of $\int_0^1 \omega$, $\int_1^{\frac{1}{\lambda}} \omega$, $\int_1^\infty \omega$ etc. One may realise these periods as hypergeometric functions.

$$\int_0^1 \omega = \int_0^1 \frac{dx}{y} = \int_0^1 \frac{dx}{\sqrt{x(1-x)(1-\lambda x}}$$
$$= \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}(1-\lambda x)^{-\frac{1}{2}}$$

from equation (4) which is simply

$$= \frac{\Gamma(\frac{1}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(1)} {}_2F_1\left(\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{array}\middle|\lambda\right) = \pi \cdot {}_2F_1\left(\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{array}\middle|\lambda\right)$$

We conclude this section by looking at another example for which we use such integration of differential forms to compute period matrix and verify Deligne's conjecture.

Example 4.4. Consider the curve

$$C := y^2 = 1 - x^5 \text{ over } \mathbb{Q}$$

(This type of curves are called Complex Multiplication.) To include the point at infinity we make a transformation $x = \frac{1}{\xi}$ and multiply both sides by ξ^6 to get

$$y^2\xi^6 = \xi^6 - \xi$$
$$\implies (y\xi^3)^2 = \xi^6 - \xi$$
$$\implies \eta^2 = \xi^6 - \xi \text{ where } \eta = y\xi^3$$

Rewriting, $C := y^2 = x^6 - x$
the 1-forms are : $\omega_1 = \frac{dx}{y}$ and $\omega_2 = \frac{xdx}{y}$
Now $\lambda = \int_0^1 \omega_1$ and $\mu = \int_0^1 \omega_2$ gives periods. Let ζ be a primitive 5th root of unity. Then

$$\int_0^\zeta \frac{dx}{\sqrt{x^6 - x}} = \int_0^1 \frac{d(t\zeta)}{\sqrt{\zeta^6 t^6 - t\zeta}}$$

since $\zeta^5 = 1$, we get

$$\frac{\zeta}{\sqrt{\zeta}} \int_0^1 \frac{dt}{\sqrt{t^6 - t}} = \zeta^{-\frac{1}{2}} \int_0^1 \frac{dt}{\sqrt{t^6 - t}} = \zeta^{-\frac{1}{2}}\lambda$$

In the above equation we really have a choice of sign there. To make it real we take its conjugate circle i.e. $\int_0^{\zeta^{-1}} \omega_1$ into count and substract them. We do similar computations to obtain $\int_0^\zeta \omega_2 = \zeta^{-\frac{3}{2}}$. Thus the real period matrix is given by :

$$M = \begin{pmatrix} \lambda & \mu \\ (\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}})\lambda & (\zeta^{\frac{3}{2}} - \zeta^{-\frac{3}{2}})\mu \end{pmatrix}$$

Below are the computational results :

$$det(M) = 10.314071041733177562983179141216861078040501050212$$

$$L(C, 1) = 1.0314071041733177562983179141216861078043261206521$$

$$\frac{L(C,1)}{det(M)} = 0.1000000000000000000000000000000000000002676107521 \sim \frac{1}{10}$$

(pari gp was used to evaluate the integrals.)

We can express the periods in terms of what is known as **Appell-Lauricella** hypergeometric functions defined as :

$$F\left(\begin{matrix} a & b_1 & \cdots & b_n \\ & & & c \end{matrix}\middle| z_1, z_2, \cdots, z_n\right) = \sum_{i_1, i_2, \ldots, i_n = 0}^{\infty} \frac{(a)_{i_1+i_2+\ldots+i_n}(b_1)_{i_1}\cdots(b_n)_{i_n}}{(c)_{i_1+i_2+\ldots+i_n} i_1! i_2! \ldots i_n!} z_1^{i_1} \ldots z_n^{i_n}$$

the integral representation is given by,

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-z_1 t)^{-b_1} \cdots (1-z_n t)^{-b_n} dt \tag{6}$$

So for the above example:

$$\lambda = \int_0^1 \frac{dx}{\sqrt{x-x^6}} = \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}(1-\zeta x)^{-\frac{1}{2}}(1-\zeta^2 x)^{-\frac{1}{2}}(1-\zeta^3 x)^{-\frac{1}{2}}(1-\zeta^4 x)^{-\frac{1}{2}}$$

$$= \pi \cdot F\left(\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & & & & 1 \end{matrix}\middle| \zeta, \zeta^2, \zeta^3, \zeta^4\right)$$

5. Gaussian Hypergeometric Functions

An important observation for the example **(4.3)** is if we truncate this hypergeometric series,

$${}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix}\middle| \lambda\right)_p := \sum_{n=0}^{p-1} \left(\frac{(-\frac{1}{2})_n}{n!}\right)^2 \lambda^n = H_p(\lambda) \text{ (say)}$$

This is called the Hasse invariant. Then

Theorem 5.1.

$$a_p(\lambda) \equiv (-1)^{\frac{p-1}{2}} H_p(\lambda) \ (mod\ p)$$

Where $a_p(\lambda) = p + 1 - N_p$, N_p being the number of points on E_λ over $\mathbb{F}_p$.

Some of such results on truncation of hypergeometric functions is studied in the paper [1].

Considering its importance we looked at finite field analogues of such hypergeometric functions called the Gaussian hypergeometric functions. Many results in which is due to Greene[2]. We also apparently came up with a new result.

Definition 5.2. *Let $\chi \in \widehat{\mathbb{F}_q^*}$ be a multiplicative character. We extend it to all $\mathbb{F}_q$ by setting $\chi(0) = 0$. Let Tr be trace map form $\mathbb{F}_q$ to $\mathbb{F}_p$. Let ζ be a primitive p-th root of unity. Then the Gauss sum is defined as :*

$$G(\chi) = \sum_{t \in \mathbb{F}_q} \chi(t) \zeta^{Tr(t)}$$

Definition 5.3. *Let $A, B \in \widehat{\mathbb{F}_q^*}$ by convention we set $A(0) = 0$. Then the Jacobi sum is defined as :*

$$J(A,B) = \sum_{z \in \mathbb{F}_q} A(z) B(1-z)$$

Greene introduces a binomial symbol:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{B(-1)}{q} J(A, \overline{B})$$

Having defined it this way one can prove certain finite field analogues of identities satisfied by hypergeometric functions. The Gauss sum defined above is analogous to the classical $\Gamma-$function. Even an analogue for the the integral formula as in **(4)** is as follows :

$${}_2F_1\left(\begin{matrix} A & B \\ & C \end{matrix}\middle| z\right)_p = \epsilon(x)\frac{BC(-1)}{q}\sum_{y\in\mathbb{F}_q} B(y)\overline{B}C(1-y)\overline{A}(1-zy)$$

where $\epsilon(x)$ is the trivial character.

Greene also gives a generalised definition for ${}_{r+1}F_r$-type hypergeometric functions and proves analogous identities.

But everything in Green's paper [2] is restricted to one parameters. What we do is try and come up with finite field analogues of Appel-Lauricella hypergeometric functions as in **(6)** and try to prove a similar identity for the integral form as in **(7)**.

Definition 5.4. *We define the* ***finite field Appell-Lauricella series*** *to be*

$$F\left(\begin{matrix} A & B_1 & \cdots & B_n \\ & & & C \end{matrix}\middle| z_1, \ldots, z_n\right) = \frac{q^n}{(q-1)^n}\sum_{\chi_1,\ldots,\chi_n\in\widehat{\mathbb{F}_q^*}} \binom{A\chi}{C\chi}\binom{B_1\chi_1}{\chi_1}\cdots\binom{B_n\chi_n}{\chi_n}\chi_1(z_1)\cdots\chi_n(z_n), \tag{7}$$

where $\chi = \prod_i \chi_i$ *and* $A, B_i, C \in \widehat{\mathbb{F}_q^*}$*, with* $z_i \in \mathbb{F}_q^*$.

Theorem 5.5. *(Fourier Transform)*
Let $g : \left(\mathbb{F}_q^*\right)^n \to \mathbb{F}_q$*. Then*

$$g(z_1, \ldots, z_n) = \sum_{\chi_1,\ldots,\chi_n\in\widehat{\mathbb{F}_q^*}} \hat{g}(\chi_1, \ldots, \chi_n)\chi_1(z_1)\ldots\chi_n(z_n),$$

where $\hat{g}$ *is given by*

$$\hat{g}(\chi_1, \ldots, \chi_n) = \frac{1}{(q-1)^n}\sum_{y_1,\ldots,y_n\in\mathbb{F}_q^*} g(y_1, \ldots, y_n)\overline{\chi}_1(y_1)\cdots\overline{\chi}_n(y_n) \tag{8}$$

is the ***Fourier transform*** *of* g.

Conjecture :
The finite field Appell-Lauricella series can be written as

$$F\left(\begin{matrix} A & B_1 & \cdots & B_n \\ & & & C \end{matrix}\middle| z_1, \ldots, z_n\right) = K\cdot\epsilon(z_1\cdots z_n)\sum_{y\in\mathbb{F}_q} A(y)\bar{A}C(1-y)\overline{B}_1(1-z_1y)\cdots\overline{B}_n(1-z_ny). \tag{9}$$

where K is some constant to be determined.

Proof. We use Fourier transforms. First, fix A, B_i, C and let $g(z_1, \ldots, z_n)$ denote the RHS of (9). Then examining the RHS of (9), we see that it suffices to show

$$\hat{g}(\chi_1, \ldots, \chi_n) = \frac{q^n}{(q-1)^n}\binom{A\chi}{C\chi}\binom{B_1\chi_1}{\chi_1}\cdots\binom{B_n\chi_n}{\chi_n}. \tag{10}$$

We first simplify the RHS of (11). We have

$$\binom{A\chi}{C\chi}\binom{B_1\chi_1}{\chi_1}\cdots\binom{B_n\chi_n}{\chi_n} = \frac{C\chi(-1)}{q}J(A\chi, C\overline{\chi})\frac{\chi_1(-1)}{q}J(B_1\chi_1,\overline{\chi}_1)\cdots\frac{\chi_n(-1)}{q}J(B_n\chi_n,\overline{\chi}_n)$$

$$= \frac{C(-1)}{q^{n+1}}\sum_{y\in\mathbb{P}^1(\mathbb{F}_q)} A\chi(y)\overline{C\chi}(1-y)\prod_{i=1}^{n}\left(\sum_{t_i\in\mathbb{P}^1(\mathbb{F}_q)} B_i\chi_i(t_i)\overline{\chi}_i(1-t_i)\right).$$

$$= \frac{C(-1)}{q^{n+1}}\sum_{y\in\mathbb{P}^1(\mathbb{F}_q)} A\chi(y)\overline{C\chi}(1-y)\prod_{i=1}^{n}\left(\sum_{t_i\in\mathbb{P}^1(\mathbb{F}_q)} B_i(t_i)\chi_i\left(\frac{t_i}{1-t_i}\right)\right), \tag{11}$$

where $\chi = \prod_i \chi_i$.
Returning to $\hat{g}$, let $z = \prod_i z_i$, we use (8) to get
$\hat{g}(\chi_1,\ldots,\chi_n)$

$$= \frac{K}{(q-1)^n}\sum_{z_1,\ldots,z_n\in\mathbb{P}^1(\mathbb{F}_q)}\epsilon(z)\sum_{y\in\mathbb{P}^1(\mathbb{F}_q)} A(y)\overline{A}C(1-y)\overline{B}_1(1-z_1y)\cdots\overline{B}_n(1-z_ny)\overline{\chi}_1(z_1)\cdots\overline{\chi}_n(z_n)$$

$$= \frac{K}{(q-1)^n}\sum_{y\in\mathbb{P}^1(\mathbb{F}_q)} A(y)\overline{A}C(1-y)\prod_{i=1}^{n}\left(\sum_{z_i\neq 0}\overline{B}_i(1-z_iy)\overline{\chi}_i(z_i)\right).$$

We now change variables by $t_i = \frac{1}{1-z_iy}$. This means $\overline{B}_i(1-z_iy) = B(t_i)$. From $\frac{1}{z_i} = \frac{yt_i}{t_i-1}$, we get

$$\overline{\chi}_i(z_i) = \chi_i\left(\frac{yt_i}{t_i-1}\right) = \chi_i(y)\chi_i\left(\frac{t_i}{t_i-1}\right).$$

Putting all this into the expression for $\hat{g}$, we get

$$\hat{g}(\chi_1,\ldots,\chi_n) = \frac{K}{(q-1)^n}\sum_{y\in\mathbb{P}^1(\mathbb{F}_q)} A(y)\overline{A}C(1-y)\chi(y)\prod_{i=1}^{n}\left(\sum_{t_i\in\mathbb{P}^1(\mathbb{F}_q)} B_i(t_i)\chi_i\left(\frac{t_i}{t_i-1}\right)\right).$$

$$= \frac{K\chi(-1)}{(q-1)^n}\sum_{y\in\mathbb{P}^1(\mathbb{F}_q)} A(y)\overline{A}C(1-y)\chi(y)\prod_{i=1}^{n}\left(\sum_{t_i\in\mathbb{P}^1(\mathbb{F}_q)} B_i(t_i)\chi_i\left(\frac{t_i}{1-t_i}\right)\right) \tag{12}$$

where again we set $\chi = \prod_i \chi_i$. Comparing (11) and (12), we must only show that

$$\frac{q^nC(-1)}{q^{n+1}(q-1)^n}\sum_{y\in\mathbb{P}^1(\mathbb{F}_q)} A\chi(y)\overline{C\chi}(1-y) = \frac{K\chi(-1)}{(q-1)^n}\sum_{t\in\mathbb{P}^1(\mathbb{F}_q)} A(t)\overline{A}C(1-t)\chi(t). \tag{13}$$

i.e.

$$\frac{C(-1)}{q}\sum_{y\in\mathbb{P}^1(\mathbb{F}_q)} A\chi(y)\overline{C\chi}(1-y) = K\chi(-1)\sum_{t\in\mathbb{P}^1(\mathbb{F}_q)} A(t)\overline{A}C(1-t)\chi(t). \tag{14}$$

Simplifying the LHS of (14) gives

$$\frac{C(-1)}{q}\sum_{y\in\mathbb{P}^1(\mathbb{F}_q)} A\chi(y)\overline{C\chi}(1-y) = \frac{C(-1)}{q}\sum_{y\in\mathbb{P}^1(\mathbb{F}_q)} A(y)C\left(\frac{1}{1-y}\right)\chi\left(\frac{y}{1-y}\right), \tag{15}$$

and simplifying the RHS of (14) gives

$$K\chi(-1)\sum_{t\in\mathbb{P}^1(\mathbb{F}_q)} A(t)\overline{A}C(1-t)\chi(t) = K\chi(-1)\sum_{t\in\mathbb{P}^1(\mathbb{F}_q)} A\left(\frac{t}{1-t}\right)C(1-t)\chi(t). \tag{16}$$

Changing variables by $t\mapsto -t$ in (16) gives

$$K\chi(-1)\sum_{t\in\mathbb{P}^1(\mathbb{F}_q)} A\left(\frac{t}{1-t}\right)C(1-t)\chi(t) = KA(-1)\sum_{t\in\mathbb{P}^1(\mathbb{F}_q)} A\left(\frac{t}{1+t}\right)C(1+t)\chi(t). \tag{17}$$

Finally, making the change of variables $t=\frac{y}{1-y}$ (equivalently $y=\frac{t}{1+t}$) in (17) gives

$$KA(-1)\sum_{t\in\mathbb{P}^1(\mathbb{F}_q)} A\left(\frac{t}{1+t}\right)C(1+t)\chi(t) = KA(-1)\sum_{y\in\mathbb{P}^1(\mathbb{F}_q)} A(y)C\left(\frac{1}{1-y}\right)\chi\left(\frac{y}{1-y}\right),$$

which is precisely the RHS of (15) if we set $K=\frac{AC(-1)}{q}$ □

So we have the following theorem :

Theorem 5.6.

$$F\left(\begin{matrix} A & B_1 & \cdots & B_n \\ & & & C\end{matrix}\,\middle|\, z_1,\ldots,z_n\right) = \frac{AC(-1)}{q}\cdot\epsilon(z_1\cdots z_n)\sum_{y\in\mathbb{F}_q} A(y)\bar{A}C(1-y)\overline{B}_1(1-z_1y)\cdots\overline{B}_n(1-z_ny).$$

6. Acknowledgments

I would like to thank Prof. Jerome W. Hoffman of Louisiana State University for his guidance and constant support in this project. I was supported by a grant from the Indo-US Science and Technology Forum (IUSSTF) under S. N. Bose Scholarship and from the Math-REU 2015, Louisiana State University and it is a pleasure to thank them for their generosity. Lastly I would like to thank Adele Zhou, Benjamin Anzis and Jenifer Lee, my other REU mates for their helpful discussions and comments. `Magma` and `Sage` were mostly used, for our computations related to this project.

7. Appendix - A

Output of example **3.1** using the inbuilt function in magma :

```
[zet1(n) : n in [5..150] | IsPrime(n) ]

[
[ 5, 0, -3, 0 ],
[ 7, -3, -6, 43 ],
[ 11, 0, 6, 0 ],
[ 13, -3, 12, -47 ],
[ 17, 0, -3, 0 ],
[ 19, 6, -15, -200 ],
[ 23, 0, 42, 0 ],
[ 29, 0, 33, 0 ],
[ 31, 3, -12, 25 ],
[ 37, 6, 30, 358 ],
[ 41, 0, -3, 0 ],
[ 43, 3, -30, -389 ],
[ 47, 0, 78, 0 ],
[ 53, 0, -3, 0 ],
[ 59, 0, 42, 0 ],
[ 61, -3, 12, -533 ],
[ 67, -3, -6, 583 ],
[ 71, 0, 78, 0 ],
[ 73, 15, 120, 1105 ],
[ 79, -3, 138, -137 ],
[ 83, 0, -3, 0 ],
[ 89, 0, 159, 0 ],
[ 97, 6, -33, 52 ]
[ 101, 0, 231, 0 ],
[ 103, 6, 93, 1312 ],
[ 107, 0, -84, 0 ],
[ 109, 3, -102, -695 ],
[ 113, 0, 195, 0 ],
[ 127, 12, -3, -2072 ],
[ 131, 0, 186, 0 ],
[ 137, 0, 141, 0 ],
[ 139, -21, 264, -2135 ],
[ 149, 0, 177, 0 ]
]
```

8. Appendix - B

The output of the example **4.1** is given below along with the code written in `Magma`.

```
P<x> := PolynomialRing(Integers());

C1 := HyperellipticCurve( x^6-14*x^5+61*x^4-106*x^3+66*x^2-8*x-3); C1;

Hyperelliptic Curve defined by y^2 = x^6 - 14*x^5 + 61*x^4 - 106*x^3 +
66*x^2 - 8*x - 3 over Rational Field


Conductor(C1)
961

L:= LSeries(C1)
Sp :=Evaluate(L,1); Sp

0.449287723876040786113329668188


C<I> := ComplexField(50);
R<x> := PolynomialRing(C);
f1:= x^6-14*x^5+61*x^4-106*x^3+66*x^2-8*x-3; f1

x^6 - 14.000000000000000000000000000000000000000000000000*x^5 +
61.000000000000000000000000000000000000000000000000*x^4 -
106.00000000000000000000000000000000000000000000000*x^3 +
66.000000000000000000000000000000000000000000000000*x^2 -
8.0000000000000000000000000000000000000000000000000*x -
3.0000000000000000000000000000000000000000000000000


A1 := AnalyticJacobian(f1) ; A1

Analytic Jacobian of the hyperelliptic curve defined by y^2 = x^6 -
14.000000000000000000000000000000000000000000000000*x^5 +
61.000000000000000000000000000000000000000000000000*x^4 -
106.00000000000000000000000000000000000000000000000*x^3 +
66.000000000000000000000000000000000000000000000000*x^2 -
8.0000000000000000000000000000000000000000000000000*x -
3.0000000000000000000000000000000000000000000000000 over Complex field
of precision 50

M1:= BigPeriodMatrix(A1); M1

[-0.76965110808045461237496888101583970284939544827801 -
2.2868326219178581582338078236741664752637308033310*I
0.73969084944133416177055129411900632636900509843295 -
0.55767424871489916880323153485854981346112473151680*I
```

```
-1.3482951308280612327081107305413816651897761192623 +
6.9500740446536324110568293491705142296497784169705E-50*I
0.51310073872030307491664592067722646856626363218419 -
6.1217716654627016108041592079835720586631073088875E-50*I]
[2.0224426962420918490621660958120724977846641788920 -
2.8445068706327573270370393585327162887248555348441*I
4.623529415231790318457474041149686957909709028772 2 -
2.2868326219178581582338078236741664752637308033212*I
-0.83519439210775815779146480986415519662351248707813 -
3.2073176521106347753686437605817782545782362561261E-50*I
-1.3482951308280612327081107305413816651897761192620 +
2.2718500035783662992194559970787595969929173480893E-50*I]


X := Matrix(C, 2, 4, [Real(M1[i][j]) : j in [1..4], i in [1..2]]);
Y := Matrix(C, 2, 4, [Im(M1[i][j]) : j in [1..4], i in [1..2]]);
S := BlockMatrix(2,1, [X, Y]);
T := BlockMatrix(2,1, [X, -Y]);
F := S^-1*T;
expr := F^2; expr

[0.99999999999999999999999999999999999999999999999999
-4.6769003371410090597704336478383946210649966100060E-52
1.0718830837300702358082987851767025189001483995098E-99
-1.3715848695387299786589402579664091513422919937874E-99]
[1.2543279018254735084461328657528269071410073458129E-50
1.0000000000000000000000000000000000000000000000000
-7.9919668229200433459295566054271767436332583587829E-100
1.8287798260516399715452536772885455351230559917165E-99]
[2.5444047524244225190189215296055093076448736374121E-50
-2.1739607168472320190390849718210409964150164498616E-50
1.0000000000000000000000000000000000000000000000000
-4.4194193312414150423314815398417929795557507912624E-51]
[0.00000000000000000000000000000000000000000000000000
-1.2829270608442539101474575042327113018312945024504E-49
1.2550203431434793094533482155747753272003328792452E-50
1.0000000000000000000000000000000000000000000000000]


seq := [1: i in [1..4]];
Id := DiagonalMatrix(C, 4, seq);
Z := F+Id;
K1 := Matrix(C, 4, 4, [Round(Z[i][j])/2 : j in [1..4], i in [1..4]]);
N :=M1*K1; N


[-0.76965110808045461237496888101583970284939544827628 +
9.1826574981940524162062388119753580879946609633314E-50*I
0.73969084944133416177055129411900632636900509843297 +
4.8793180966763054104251539962031588021831006467631E-50*I
```

```
-1.348295130828061232708110730541381665189776119262 3 +
6.9500740446536324110568293491705142296497784169705E-50*I
0.5131007387203030749166459206772264685662636321841 9 -
6.1217716654627016108041592079835720586631073088875E-50*I]
[2.0224426962420918490621660958120724977846641788930 -
3.4077750053675494488291839956181393954893760221339E-50*I
4.6235294152317903184574740411496869579097090287722 -
8.6864853077996358499567435182423161061493898603414E-51*I
-0.83519439210775815779146480986415519662351248707813 -
3.2073176521106347753686437605817782545782362561261E-50*I
-1.3482951308280612327081107305413816651897761192620 +
2.2718500035783662992194559970787595969929173480893E-50*I]

%We see that Column 1 and 3 are linearly independent

list := [Real(Ntr[1][1]), Real(Ntr[1][2]), Real(Ntr[3][1]),
   Real(Ntr[3][2])];
W := Matrix(C, 2, 2, list); W

[-0.76965110808045461237496888101583970284939544827628
2.0224426962420918490621660958120724977846641788930]
[-1.3482951308280612327081107305413816651897761192623
-0.83519439210775815779146480986415519662351248707813]


Delta:=Determinant(W); Delta

3.3696579290703058958499725114094293500583517690928

Sp/Delta

0.133333333333333333333333333333
```

REFERENCES

[1] Alyson Deines, Jenny G. Fuselier, Ling Long, Holly Swisher, Fang-Ting Tu, *Hypergeometric Series, Truncated Hypergeometric series, and Gaussian Hypergeometric Functions*, arXiv:1501.03564v1, Jan 2015.

[2] John Greene, *Hypergeometric functions over finite fields*, American Mathematical Society, Volume 301, Number 1, May 1987.

[3] Dermot McCarthy, *TRANSFORMATIONS OF WELL-POISED HYPERGEOMETRIC FUNCTIONS OVER FINITE FIELDS*, arXiv:1204.4377v2

[4] Ira Gessel, Dennis Stanton, *Strange Evaluations of Hypergeometric series*, SIAM J. MATH. ANAL.,vol. 13, No. 2, March 1982

[5] Michel Borner, Irene I. Bouw, and Stefan Wewers, *The functional equation for L-functions of hyperelliptic curves*, arXiv:1504.00508v1,

[6] Irene I. Bouw, Stefan Wewers, *COMPUTING L-FUNCTIONS AND SEMISTABLE REDUCTION OF SUPERELLIPTIC CURVES*, arXiv:1211.4459v4

[7] Holzapfel, Rolf-Peter, Nicolae, Florin, *Arithmetic on a family of Picard curves*, ISBN: 9783540439615, Vol. 2002,

[8] Kai Arzdorf, Stefan Wewers, *Another proof of the Semistable Reduction Theorem*, arXiv:1211.4624v1

E-mail address: samprit.ghosh.1993@gmail.com

DEPARTMENT OF MATHEMATICS, IIT BOMBAY, MUMBAI-400076

MATHEMATICS REU-2015, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LA-70803